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Flat maximal immersions

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Abstract

This paper proves that a complete maximal Lorentzian surface in three-dimensional Minkowski space is flat iff it is the graph over a timelike plane of a function of one variable. Furthermore, we prove that maximal Lorentzian immersions are always unstable. Finally, we find the maximal Lorentzian surfaces of revolution.

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0. Introduction

The study of minimal surfaces has a long history going back to the experiments of the Belgian physicist J. Plateau in 1847. He experimented with a wire dipped in soap film and showed that the surface formed was stable with respect to area, that is to say slight deformations of the surface increased the area. Later mathematicians showed that the surfaces involved were minimal surfaces, i.e. surfaces with mean curvature zero. One can prove that for small domains in the surface the area functional has a local minimum. This paper answers the corresponding problem in relativity by considering Lorentzian surfaces of zero mean curvature in Minkowski space. We prove that these surfaces neither maximize nor minimize area.

There are several interesting papers related to the present paper, see for instance [1–5,7].

1. Flat maximal surfaces in $\mathbb{R}^3_1$

The metric on $\mathbb{R}^3_1$ is

$$\mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}x_3^2.$$

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We start with the following lemma.

Lemma 1.1. *Let (M, g) denote a connected, oriented, real analytic, complete maximal surface in $\mathbb{R}^3_1$. Then the following two conditions are equivalent:*

(1) *(M, g) is flat.*

(2) *There exists a timelike plane Π_* in $\mathbb{R}^3_1$ with unit normal vector v, a real analytic function $f : I \to \mathbb{R}$ defined on an open interval I and a linear function $L : \Pi_* \to \mathbb{R}$ such that*

$$M = \{x + h(x)v \mid x \in L^{-1}(I)\}, \quad h = f \circ L_{|L^{-1}(I)}.$$

Remark 1.2. Briefly (M, g) is flat iff it is the graph of a function $h = f \circ L_{|L^{-1}(I)}$.

Proof. To see that (2) implies (1), let A denote an isometry of $\mathbb{R}^3_1$ such that

$$A(\Pi_*) = \Pi = \{(x_1, x_2, 0) \in \mathbb{R}^3_1 \mid (x_1, x_2) \in \mathbb{R}^2_1\}, \quad A(v) = (0, 0, 1).$$

Then

$$N = A(M) = \{(x_1, x_2, k(x_1, x_2)) \mid x \in A(L^{-1}(I)) = U\}, \quad k = f \circ L \circ A^{-1}_{|U}.$$

Now

$$L \circ A^{-1}_{|U}(x_1, x_2, 0) = a_1 x_1 + a_2 x_2$$

and

$$k_{11} = f'' a_1^2, \qquad k_{12} = f'' a_1 a_2, \qquad k_{22} = f'' a_2^2,$$

hence

$$K = (k_{11} k_{22} - k_{12}^2)/* = 0.$$

To prove that (1) implies (2) let (M, g) be flat and

$$G : M \to S^2_1$$

denote the Gauss map. Let A denote an orientation preserving isometry of $\mathbb{R}^3_1$ such that

$$A(G(p)) = (0, 0, 1)$$

for some $p \in M$. Let $N = A(M)$ and

$$H : N \to S^2_1$$

denote the Gauss map of N. According to [6]

$$\operatorname{Im} H \subset \{(t, \pm t, 1) \mid t \in \mathbb{R}\}.$$

It follows that

$$T_q\pi : T_q N \to T_{\pi(q)}\Pi, \quad \pi : N \to \Pi, \ (x_1, x_2, x_3) \mapsto (x_1, x_2, 0)$$

is an isomorphism for every $q \in N$.

Define stereographic projection

$$\sigma : S_1^2 \backslash \{q_3 = 1\} \to A, \quad (q_1, q_2, q_3) \mapsto (q_1/(1-q_3), \ -q_2/(1-q_3)).$$

According to [6] there are real analytic $\mathbb{L}$ differentiable functions f and $g : V \to \mathbb{L}$ such that

$$\sigma \circ \pm H \circ \Phi_{(f,g)}(z) = g(z),$$

where

$$\Phi_{(f,g)} : V \to N, \quad \Phi_{(f,g)}(z_0) = p, \ z_0 \in V$$

is an isothermal coordinate system on N around p defined by [6, 7.3].

There are now two possibilities.

(1) g constant, hence H constant. (2) g nonconstant. We can then assume $g'(z_0) = (a, \pm a)$ is null, hence nonzero.

Now let σ denote the complete null geodesic with initial velocity

$$v = \begin{cases} (1,1) & \text{in case (1)}, \\ (a, \mp a) & \text{in case (2)} \end{cases}$$

in the coordinate system $\Phi_{(f,g)}$.

Let δ denote a null vector field along σ in TN such that $\sigma'(s)$ and $\delta(s)$ are linearly independent.

Let τ_s denote the complete null geodesic with initial velocity $\delta(s)$. Since g' is everywhere null or zero

$$\tau_s'(t) \in \ker T_{\tau_s(t)} H$$

for small s and t, hence $\tau_s'(t)$ lies in the two plane through $\tau_s(0)$ orthogonal to $H(\tau_s(0))$.

Hence τ_s can be reparametrized to a null geodesic of $\mathbb{R}_1^3$ for all $s \in \mathbb{R}$. So

$$\beta_s(t) = \sigma(s) + t\delta(s) \in N$$

for all $s, t \in \mathbb{R}$. We claim that

$$N \subset \operatorname{Im} \beta.$$

Assume for contradiction that there exists a $q \in N \backslash \operatorname{Im} \beta$. Since N is connected there exists a $q^* \in \partial \operatorname{Im} \beta$. Since $q^* \in N$ we can define as above a null geodesic σ_* through q^* and a null vector field δ_* along σ_* such that $\sigma_*'(s)$ and $\delta_*(s)$ are linearly independent:

$$(s,t) \mapsto \sigma_*(s) + t\delta_*(s)$$

is a chart on N near $(0,0)$. Maximality implies that

$$\delta_*'(s) \in \operatorname{span}\{\sigma_*'(s), \delta_*(s)\}$$

while nondegeneracy of the metric induced by β implies that

$$\langle \delta_*'(s), \delta_*(s) \rangle = 0$$

so $\delta_*'(s)$, the covariant derivative of δ_* in $\mathbb{R}^3_1$, and $\delta_*(s)$ are colinear. We can then assume that δ_* is constant.

We can then also assume δ is constant. By connectedness of N we can assume $\delta = \delta_*$. Now take a sequence $\{(s_n, t_n)\}_{n\in\mathbb{N}}$ in $\mathbb{R} \times \mathbb{R}$ such that

$$q^* = \lim_{n\to+\infty} \beta(s_n, t_n).$$

We can now take (s_*, t_*), $(s, t) \in \mathbb{R} \times \mathbb{R}$ such that

$$\beta_*(s_*, t_*) = \sigma_*(s_*) + t_*\delta = \sigma(s) + t\delta = \beta(s, t).$$

Notice that

$$\sigma_1(u) = \sigma(u) + (t - t_*)\delta$$

is a null curve in N. Define

$$\beta_1(u, v) = \sigma_1(u) + v\delta.$$

Then

$$\beta_1(s, t_*) = \sigma_1(s) + t_*\delta = \sigma(s) + t\delta = \sigma_*(s_*) + t_*\delta.$$

Since σ_1 and σ_* are both null curves on N through $\sigma_1(s) = \sigma_*(s_*)$ such that

$$\sigma_1'(s), \qquad \sigma_*'(s_*)$$

are linearly dependent we can reparametrize σ_1 to σ_* with a smooth bijection $h : \mathbb{R} \to \mathbb{R}$ as

$$\sigma_1 = \sigma_* \circ h.$$

Now

$$\beta_*(u, v) = \beta(h^{-1}(u), v + t - t_*).$$

Hence

$$\beta_*(0, 0) = q^* \in \operatorname{Im} \beta.$$

Now β_* is onto an open neighbourhood of q^* and hence so is β. This contradicts $q^* \in \partial \operatorname{Im} \beta$ and shows that $N = \operatorname{Im} \beta$. We claim that

$$\tau(s, t) = (\sigma_1(s) + t\delta_1, \sigma_2(s) + t\delta_2) = \pi \circ \beta(s, t), \qquad s, t \in \mathbb{R}$$

is injective.

Choose a basis $e_1 = \pi(\delta)$, e_2 in $\mathbb{R}^2$. Notice that $\pi(\sigma'(s))$ and $\pi(\delta)$ are linearly independent, because $T_q\pi$ is an isomorphism for all q and $\sigma'(s)$ and δ are linearly independent in T_qN. Writing

$$(\sigma_1(s), \sigma_2(s)) = \eta_1(s)e_1 + \eta_2(s)e_2,$$

we see that $\eta_2'(s)$ is either everywhere positive or everywhere negative. This proves the claim. So

$$\pi : N \to \pi(N) = U$$

has an inverse

$$\rho : U \to N, \quad \rho(x) = (x_1, x_2, \rho_3(x)).$$

Now define

$$U_* = A^{-1}(U), \qquad \Pi_* = A^{-1}(\Pi), \qquad h = \rho_3 \circ A_{|\Pi_*}.$$

Then

$$M = \{x + h(x)v \mid x \in U_* \subset \Pi_*\}.$$

Notice that

$$\frac{\mathrm{d}}{\mathrm{d}s}\rho_3(x_0 + s(1, \mp 1)) \equiv 0$$

because $\beta(s,t) = \sigma(s) + t\delta \in N$ and $\delta = (\alpha, \mp\alpha, 0)$, $\alpha \in \mathbb{R}\backslash\{0\}$. Define

$$f(s) = \rho_3(s(1, \pm 1)), \quad s \in I,$$
$$L(x) = \tfrac{1}{2}(x_1 \pm x_2).$$

Then

$$\begin{aligned} f \circ L(x_1, x_2) &= \rho_3(\tfrac{1}{2}(x_1 \pm x_2)(1, \pm 1)) \\ &= \rho_3(\tfrac{1}{2}(x_1 \pm x_2)(1, \pm 1) + \tfrac{1}{2}(x_1 \mp x_2)(1, \mp 1)) \\ &= \rho_3(x_1, x_2), \end{aligned}$$

which proves the lemma. □

We can improve the lemma to the following theorem.

Theorem 1.3. *Let (M, g) denote a connected, oriented, real analytic, complete maximal surface in $\mathbb{R}^3_1$. Then the following two conditions are equivalent:*
(1) *(M, g) is flat.*
(2) *There exists a timelike plane Π_* in $\mathbb{R}^3_1$ with unit normal vector v, a real analytic function $f : \mathbb{R} \to \mathbb{R}$ and a linear function $L : \Pi_* \to \mathbb{R}$ such that*

$$M = \{x + h(x)v \mid x \in \Pi_*\}, \quad h = f \circ L.$$

Proof. We have already seen that (2) implies (1). If (M, g) is flat Lemma 1.1 guarantees the existence of $v, L : \Pi_* \to \mathbb{R}$ and $f : I \to \mathbb{R}$ as in Lemma 1.1. (2) Let $v_1, v_2 = v, v_3$ denote an orthonormal basis in $\mathbb{R}^3_1$ where v_1, v_3 is an orthonormal basis in Π_*. In coordinates with respect to this orthonormal basis the surface is the image of the map

$$\rho(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ f(a_1x_1 + a_2x_2) \end{pmatrix},$$

where $a_1, a_2 \in \mathbb{R}$. The coordinates of the second fundamental form in this chart are

$$l = a_1^2 f''/((a_1^2 - a_2^2)f'^2 + 1)^{1/2},$$
$$m = a_1a_2 f''/((a_1^2 - a_2^2)f'^2 + 1)^{1/2},$$
$$n = a_2^2 f''/((a_1^2 - a_2^2)f'^2 + 1)^{1/2},$$

while the metric tensor is

$$\begin{pmatrix} 1 + a_1^2 f'^2 & a_1a_2 f'^2 \\ a_1a_2 f'^2 & -1 + a_2^2 f'^2 \end{pmatrix}.$$

By [8, Vol. 3, p.198] the mean curvature H is

$$H = -(a_2^2 - a_1^2)f''/2((a_1^2 - a_2^2)f'^2 + 1)^{3/2},$$

so either

$$a_1 = \pm a_2$$

or

$$f'' = 0.$$

If $f'' = 0$ then M is a plane and by completeness $I = \mathbb{R}$. If $a_1 = a_2$ define

$$M(u, v) = \frac{1}{2}\begin{pmatrix} u + v \\ u - v \end{pmatrix}.$$

Then

$$\tau(u, v) = \rho(M(u, v)) = \begin{pmatrix} \frac{1}{2}(u + v) \\ \frac{1}{2}(u - v) \\ g(u) \end{pmatrix}$$

for some real analytic function g. In this chart the metric tensor is

$$\begin{pmatrix} g'^2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

hence

$$\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^1_{22} \equiv 0.$$

The first coordinate of a geodesic β is then

$$\beta_1(t) = b_1t + c_1, \quad b_1, c_1 \in \mathbb{R},$$

which implies that $I = \mathbb{R}$. Similarly $a_1 = -a_2$ implies $I = \mathbb{R}$ and the theorem follows. □

2. Instability of maximal immersions

Let (M, g) denote a real analytic maximal surface in $\mathbb{R}^3_1$. Also let D denote a domain in M with compact closure.

Definition 2.1. D is $+(-)$ stable if the second variation of the induced area is nonpositive (nonnegative) for all variations that leave the boundary ∂D of D fixed. Otherwise D is $+(-)$ unstable.

Let K denote the sectional curvature function on M. We start with the following lemma.

Lemma 2.2. *Let n denote a unit normal vector field on an open neighbourhood of some $p \in M$ and S_n the shape operator derived from n. Then*
(1) *$K(p) \le 0$ iff $S_n(p)$ has only real eigenvalues λ_1, λ_2.*
(2) *$K(p) > 0$ iff $S_n(p)$ has eigenvalues $\lambda = \pm \mathrm{i}b$, $b \neq 0$.*
(3) $\operatorname{tr}\{S_n(p)^2\} = -2K(p)$.
In (1) $K(p) = \lambda_1\lambda_2$. In (2) $K(p) = b^2$.

Proof. $S_n(p)$ is a self-adjoint operator. Let v_1, v_2 denote an orthonormal basis of T_pM with v_1 timelike. Since M is a maximal surface the matrix representation of $S_n(p)$ in basis v_1, v_2 is

$$\{S_n(p)\}_{i,j} = \begin{pmatrix} a_1 & a_2 \\ -a_2 & -a_1 \end{pmatrix}.$$

Define

$$\Delta = 4(a_1^2 - a_2^2).$$

There are now two cases to consider.
(1) $\Delta \ge 0$. There are two real eigenvalues

$$\lambda_\pm = \pm\sqrt{a_1^2 - a_2^2}$$

and

$$K(p) = \lambda_1\lambda_2 = -\tfrac{1}{2}(\lambda_1^2 + \lambda_2^2) = -\tfrac{1}{2}\operatorname{tr}\{S_n(p)^2\} \le 0$$

by the Gauss equation.
(2) $\Delta < 0$. Here the eigenvalues are

$$\lambda = \pm \mathrm{i}\sqrt{a_2^2 - a_1^2} = \pm \mathrm{i}b$$

and the Gauss equation gives

$$K(p) = a_2^2 - a_1^2 = b^2, \qquad \operatorname{tr}\{S_n(p)^2\} = 2(a_1^2 - a_2^2).$$

The lemma follows. □

We can now prove the following theorem.

Theorem 2.3. *Every domain $D \subset M$ with compact closure is $+(-)$ unstable.*

Proof. Let D denote a domain in M with compact closure and take a $p \in D$. Let G denote a local Gauss map defined on an open neighbourhood W of p. There exists an isometry A of $\mathbb{R}^3_1$ such that

$$A(G(p)) = (0, 0, 1).$$

According to [6] there exist $\mathbb{L}$ differentiable real analytic functions $f, g : V \to \mathbb{L}$ such that

$$\Phi_{(f,g)} : V \to U \subset A(D)$$

is a coordinate system around $A(p)$ on $N = A(M)$. Here $\Phi_{(f,g)}$ is defined by [6, 7.3]. We have the formulas

$$\begin{aligned} &K(z) = 16\langle g', g'\rangle / \langle f, f\rangle (1 - \langle g, g\rangle)^4(z), \quad \langle f, f\rangle = -f_1^2 + f_2^2, \\ &\langle\ ,\ \rangle = \mu(\mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y), \\ &\mu(z) = -\tfrac{1}{4}\langle f, f\rangle (1 - \langle g, g\rangle)^2(z), \quad z \in V, \end{aligned} \tag{1}$$

which will be useful later. Let

$$\alpha :]-\epsilon, \epsilon[\ \times N \to N$$

denote a variation of N such that

$$\frac{\partial \alpha}{\partial t} = hn,$$

where h vanishes on $N \backslash A(D)$ and n is a vector field on N which is a unit normal vector field to N on supp h. We shall proceed to define h explicitly. First let

$$k(q) = \begin{cases} j(z) = R^2 - \alpha^2 z_1^2 - z_2^2, & q = \Phi_{(f,g)}(z) \in U,\ \alpha^2 z_1^2 + z_2^2 \le R^2, \\ 0, & \text{otherwise.} \end{cases}$$

We have chosen $R > 0$ such that $\alpha^2 z_1^2 + z_2^2 \le R^2$ implies $z \in V$. Furthermore, let k_* denote a smooth function on $\mathbb{R}$ such that

$$\begin{aligned} &k_*(t) = 0, \quad t \le 0, \\ &k_*(t) = 1, \quad t \ge 1, \\ &|\, k_*'(t) \,| \le K, \quad \forall t \in \mathbb{R} \end{aligned}$$

for some positive real number K. Then define $k_\epsilon(t) = k_*(t/\epsilon^2)$ for $\epsilon > 0$. h shall then denote the smooth function on N which is

$$h(q) = j(z) k_\epsilon(R^2 - \alpha^2 z_1^2 - z_2^2 - \epsilon^2), \quad 0 < \epsilon < R$$

when $q = \Phi_{(f,g)}(z) \in U, \alpha^2 z_1^2 + z_2^2 \le R^2$ and zero otherwise.

By [8, p.533] the second variation of area is

$$\frac{\mathrm{d}^2 A_{\alpha,R}}{\mathrm{d}t^2}(0) = \int_N (-h^2 \Sigma_2(n) - h\Delta h)\,\mathrm{d}A, \quad \Sigma_2(n) = \mathrm{tr}\{S_n^2\}.$$

According to Lemma 2.2 this is

$$\begin{aligned}\int_N (-h^2 \Sigma_2(n) - h\Delta h)\,\mathrm{d}A \\ &= \int_U (2h^2 K + \mu\langle \mathrm{grad}\, h, \mathrm{grad}\, h\rangle)\,\mathrm{d}A \\ &= \pm \int_V 8h^2 \langle g', g'\rangle/(1 - \langle g, g\rangle)^2\,\mathrm{d}z_1\,\mathrm{d}z_2 \pm \int_V (h_1^2 - h_2^2)\,\mathrm{d}z_1\,\mathrm{d}z_2,\end{aligned}$$

where

$$h_1 = \frac{\partial h}{\partial z_1}, \qquad h_2 = \frac{\partial h}{\partial z_2}.$$

Now letting $D_R = \{(z_1, z_2) \in \mathbb{L} \mid \alpha^2 z_1^2 + z_2^2 \le R^2\}$ we compute

$$\int_{D_R} (-k_1^2 + k_2^2)\,\mathrm{d}z_1\,\mathrm{d}z_2 = \int_{D_R} -k\Delta_F k\,\mathrm{d}z_1\,\mathrm{d}z_2 = -\pi \frac{\alpha^2 - 1}{\alpha} R^4,$$

where Δ_F denotes the flat Lorentzian Laplacian on $\mathbb{L}$. There exist positive constants K_1, K_2, K_3 such that

$$|\, k_1 \,| \le K_1, \qquad |\, k_2 \,| \le K_2, \qquad |\, 8\langle g', g'\rangle/(1 - \langle g, g\rangle)^2 \,| \le K_3$$

on D_R, hence

$$\left| \int_{D_R} 8k^2 \langle g', g'\rangle/(1 - \langle g, g\rangle)^2\,\mathrm{d}z_1\,\mathrm{d}z_2 \right| \le K_3 \pi R^6 / 3\alpha.$$

This means that there exists α_+, R_+ and α_-, R_- such that

$$I(\alpha, R) = \int_{D_R} (2k^2 K + \mu\langle\, \mathrm{grad}\, k,\ \mathrm{grad}\, k\rangle)\,\mathrm{d}A$$

has

$$I(\alpha_+, R_+) > 0, \qquad I(\alpha_-, R_-) < 0.$$

Taking $\epsilon > 0$ sufficiently small we conclude

$$\frac{\mathrm{d}^2 A_{\alpha_+, R_+}}{\mathrm{d}t^2}(0) > 0, \qquad \frac{\mathrm{d}^2 A_{\alpha_-, R_-}}{\mathrm{d}t^2}(0) < 0$$

and the theorem follows. □

3. Maximal surfaces of revolution

Let $f : I \to \mathbb{R}_+$ denote a smooth function on an open interval around 0 in $\mathbb{R}$. Define the surface of revolution

$$M_T = \{x(t,s) = (f(t)\cos s, t, f(t)\sin s) \mid t \in I, s \in \mathbb{R}\}$$

in $\mathbb{R}^3_1$ still with metric tensor $\mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}x_3^2$. We shall assume: (a) $|f'| < 1$ everywhere, or (b) $|f'| > 1$ everywhere. In the first case M_T is a smooth Lorentzian surface; in the second case a Riemannian surface with the induced metric. Define

$$f_0 = f(0), \qquad x_0 = f'(0).$$

Proposition 3.1. *If M_T is maximal, then*

(a) $I \subset \left]-yf_0/\sqrt{1-x_0^2}, (\pi - y)f_0/\sqrt{1-x_0^2}\right[= J$ *and*

$$f(t) = (1/D)\sin(Dt + y), \quad D = \sqrt{1-x_0^2}/f_0, \ \ y = \operatorname{Arccos} x_0.$$

(b) $I \subset \left]-y/D, +\infty\right[$ *or* $I \subset \left]-\infty, -y/D\right[$ *and*

$$f(t) = \operatorname{sign} x_0 (1/D)\sinh(Dt + y),$$

where $|x_0| = \cosh y$, $y > 0$, $D = \operatorname{sign}(x_0)\sinh y/f_0$.

Proof. Compute

$$\langle x_t, x_t\rangle = f'(t)^2 - 1, \qquad \langle x_t, x_s\rangle = 0, \qquad \langle x_s, x_s\rangle = f(t)^2$$

and in case (a)

$$\langle \Pi(\partial_t,\partial_t), n\rangle = f''(t)/\sqrt{1-f'(t)^2},$$
$$\langle \Pi(\partial_t,\partial_s), n\rangle = 0,$$
$$\langle \Pi(\partial_s,\partial_s), n\rangle = -f(t)/\sqrt{1-f'(t)^2},$$

where Π is the second fundamental form of M and n is a unit normal vector field to M. It follows that f satisfies the second-order differential equation

$$f'(t)^2 - 1 - f(t)f''(t) = 0. \tag{2}$$

(a) Define

$$f_*(t) = (1/D)\sin(Dt + y), \quad t \in J.$$

It is the maximal solution of (2), satisfying the given initial conditions. Since f gives rise to a solution of (2) it follows that $I \subset J$ and $f = f_*$ on I.

(b) It is similar. □

Let $g : I \to \mathbb{R}_+$ denote a smooth function on an open interval around 0 in $\mathbb{R}$. Define the surface

$$M_S = \{x(t,s) = (g(t)\cosh s, g(t)\sinh s, t) \mid t \in I, s \in \mathbb{R}\}$$

in $\mathbb{R}^3_1$. It is invariant under Lorentz transformations that preserve $(0,0,1)$.

Proposition 3.2. *If M_S is maximal, then*

$$g(t) = (1/D)\cosh(Dt + y), \quad g'(0) = \sinh y, \qquad D = \sqrt{g'(0)^2 + 1}\,/g(0).$$

Proof. This time the differential equation becomes

$$g(t)g''(t) - (1 + g'(t)^2) = 0. \qquad \square$$

Let $h : I \to \mathbb{R}$ denote a smooth function defined on an open interval $I \subset \mathbb{R}\backslash\{0\}$ with $h' < 0$. Define

$$M_N = \{x(t,s) = (h(t) + t - s^2 t, h(t) - t - s^2 t, -2st) \mid s \in \mathbb{R}, t \in I\}$$

which is invariant under Lorentz isometries

$$\begin{pmatrix} 1 - s^2/2 & s^2/2 & s \\ -s^2/2 & 1 + s^2/2 & s \\ -s & s & 1 \end{pmatrix}$$

preserving the null vector $(1,1,0)$.

Proposition 3.3. *If M_N is maximal, then*

$$h(t) = at^3 + b, \quad a < 0, \ b \in \mathbb{R}.$$

Proof. The differential equation becomes

$$th'' - 2h' = 0. \qquad \square$$

There is a Clairaut Theorem for Lorentz surfaces of revolution.

Proposition 3.4. *Let $x(\gamma(t)) = \beta(t)$, $t \in A$ be a geodesic on $M_T(|f'| < 1)$, M_S or M_N. Then:*

(1) *On M_T with β' timelike (spacelike):*

$$\sinh\alpha(t) f(\gamma_1(t)) = constant \qquad (\cosh\alpha(t) f(\gamma_1(t)) = constant).$$

(2) *On M_S with β' spacelike (timelike):*

$$\sinh\alpha(t) g(\gamma_1(t)) = constant \qquad (\cosh\alpha(t) g(\gamma_1(t)) = constant).$$

(3) *On M_N with β' timelike (spacelike):*

$$\sinh\alpha(t)\gamma_1(t) = constant \qquad (\cosh\alpha(t)\gamma_1(t) = constant).$$

Remark 3.5. In cases (1) and (3): If β' is timelike define α by

$$\langle \beta'/ \mid \beta' \mid, x_s/ \mid x_s \mid \rangle = \sinh\alpha. \tag{3}$$

If β' is spacelike define α by

$$\langle \beta'/ \mid \beta' \mid, x_t/ \mid x_t \mid \rangle = \sinh\alpha. \tag{4}$$

In case (2): If β' is spacelike define α by (3) and if β' is timelike define α by (4).

Proof. We have

$$g_{11} = E(t), \qquad g_{12} = 0, \qquad g_{22} = G(t).$$

Since β is a geodesic we have

$$\gamma_2'' + (G'/G)\gamma_1'\gamma_2' = 0,$$

which shows that

$$(G \circ \gamma_1)\gamma_2' = \text{ constant } = \langle \beta', x_s(\gamma(t))\rangle$$

using $g_{12} = 0$. Now use the definition of α to reach the conclusion of the proposition. □

4. Lines of curvature

Let (M, g) denote a real analytic Lorentz surface with unit normal U in a real analytic semi-Riemannian manifold (N, h). When H and K denote the mean curvature and sectional curvature of M define

$$M_+ = \{q \in M \mid H^2 - K > 0\},$$
$$M_- = \{q \in M \mid H^2 - K < 0\},$$
$$M_0 = \{q \in M \mid H^2 - K = 0\}.$$

For every $q \in M_+$ the shape operator S_q derived from U has two real eigenvalues $\lambda_+(q) > \lambda_-(q)$. They give rise to two real analytic one-dimensional distributions

$$\Delta_+ = \ker(S_q - \lambda_+(q)\mathrm{id}),$$
$$\Delta_- = \ker(S_q - \lambda_-(q)\mathrm{id}).$$

They are everywhere orthogonal since S is self-adjoint.

M_0 is the semi-umbilic set.

Proposition 4.1. *Let (M, g) have constant mean curvature and $p \in M_0$. By $\sigma_+ : I_+ \to M$ and $\sigma_- : I_- \to M$ we denote two maximal null geodesics through p with linearly independent initial velocities. Then $\sigma_+(I_+) \subset M_0$ or $\sigma_-(I_-) \subset M_0$.*

Proof. Let (V, ϕ) denote an isothermal coordinate system around p, see [6]. Define

$$E = g_{11}, \qquad F = g_{12}, \qquad G = g_{22},$$
$$l = \langle U, \Pi(\partial_1, \partial_1)\rangle, \qquad m = \langle U, \Pi(\partial_1, \partial_2)\rangle, \qquad n = \langle U, \Pi(\partial_2, \partial_2)\rangle,$$

where U is a unit normal vector field to M. Then

$$K = (ln - m^2)/(-E^2), \qquad H = (l - n)/2E.$$

Hence

$$l_y - m_x = E_y(l - n)/2E = E_y H, \qquad m_y - n_x = E_x H.$$

Using $EH = (l - n)/2$ we conclude that

$$((l + n)/2)_y - m_x = -EH_y = 0, \qquad ((l + n)/2)_x - m_y = EH_x = 0.$$

Define

$$\Phi : V \to \mathbb{L}, \quad \Phi = ((l + n)/2, m),$$

which is $\mathbb{L}$ differentiable, see [6]. Notice that

$$\langle \Phi, \Phi \rangle = E^2(-H^2 + K),$$

so Φ is null at p. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear map with matrix representation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in the standard basis on $\mathbb{R}^2$. Then

$$L \circ \Phi \circ L^{-1}(z) = (f_1(z_1), f_2(z_2))$$

for two real analytic functions f_1 and f_2 defined on small open intervals I_1 and I_2 around 0. Since $\Phi(p)$ is null $f_1(0) = 0$ or $f_2(0) = 0$. Hence

$$\sigma_+(I_+) \subset M_0 \quad \text{or} \quad \sigma_-(I_-) \subset M_0. \qquad \square$$

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